Grade 11/12 Math Circles
October 18, 2023
Digital Signal Processing - Solutions

Exercise 1

Consider an input signal $x[n]$ and the corresponding time delayed signal $x[n-n_0]$.

Use these two signals to show that the digital filter defined by

$$y[n] = \frac{1}{2}(x[n] + x[n-1])$$

is time-invariant.

Exercise 1 Solution

Since $y[n] = \frac{1}{2}(x[n] + x[n-1])$, then $y[n-n_0] = \frac{1}{2}(x[n-n_0] + x[n-n_0-1])$.

Now we want to determine the output when the input is $x[n-n_0]$. This is given by

$$\begin{aligned}\frac{1}{2}(x[n-n_0] + x[n-1-n_0]) &= \frac{1}{2}(x[n-n_0] + x[n-n_0-1]) \\ &= y[n-n_0].\end{aligned}$$

Therefore the filter is time-invariant.

Exercise 2

Consider the input signal $\delta[n]$ and the shifted signal $\delta[n+1]$.

Use these two signals as a counterexample to show that the digital filter defined by

$$y[n] = x[n^2]$$

is not time-invariant.

Exercise 2 Solution

First, consider the input signal $x_1[n] = \delta[n]$, which is equal to 1 when $n = 0$ and equal to 0 everywhere else. The output of the filter will be $y[n] = \delta[n^2] = \delta[n]$.

Now, consider the input signal $x_2[n] = \delta[n+1]$, which is equal to 1 when $n = -1$ and equal to 0 everywhere else. We see that $z[n] = \delta[n^2 + 1]$. Computing some values we find

$$\begin{aligned} z[-2] &= \delta[4+1] = \delta[5] = 0 \\ z[-1] &= \delta[1+1] = \delta[2] = 0 \\ z[0] &= \delta[0+1] = \delta[1] = 0 \\ z[1] &= \delta[1+1] = \delta[2] = 0 \end{aligned}$$

and in general $z[n] = 0$ for all n.

This is clearly not equal to $y[n+1]$ (which is equal to $\delta[n+1]$ from above), therefore this filter is not time-invariant.

Exercise 3

Consider the LTI filter defined by

$$y[n] = 2x[n] + x[n-1]$$

and the signals $z[n] = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}$ and $z[n-1] = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$. Compute $y[n]$ for the input signal $x[n] = z[n] + z[n-1]$ by

a) evaluating the filter response to $x[n]$ directly, and

b) using the superposition property.

Exercise 3 Solution

a) Letting $x[n] = z[n] + z[n-1]$ we find that

$$x[n] = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}.$$

Therefore $y[n] = 2x[n] + x[n-1]$ can be evaluated by adding together the signals $2x[n]$ and $x[n-1]$, to find

$$y[n] = \begin{bmatrix} 2 & 5 & 6 & 4 & 1 \end{bmatrix}.$$

b) We find that

$$w[n] = 2z[n] + z[n-1] = \begin{bmatrix} 2 & 3 & 3 & 1 \end{bmatrix}$$

By the superposition property, when $x[n] = z[n] + z[n-1]$ the output is $y[n] = w[n] + w[n-1]$.

Therefore

$$y[n] = \begin{bmatrix} 2 & 5 & 6 & 4 & 1 \end{bmatrix}.$$

which agrees with what we found in a).

Exercise 4

Determine $h[n]$, i.e. determine the impulse response of the filter defined by

$$y[n] = x[n] - 2x[n-1] + 3x[n-2].$$

CHALLENGE: Without doing any calculations, could you write down the impulse response of the filter $y[n] = ax[n] + bx[n-1] + cx[n-2]$, where a, b and c are constants?

Exercise 4 Solution

The impulse response of this filter is

$$\begin{aligned} h[n] &= \delta[n] - 2\delta[n-1] + 3\delta[n-2] \\ &= \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}. \end{aligned}$$

In general, the impulse response of the filter defined by $y[n] = ax[n] + bx[n-1] + cx[n-2]$ is given by

$$\begin{aligned} h[n] &= a\delta[n] + b\delta[n-1] + c\delta[n-2] \\ &= \begin{bmatrix} a & b & c \end{bmatrix}. \end{aligned}$$

Exercise 5

Evaluate the convolution of

$$a[n] = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$$

with

$$b[n] = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix},$$

i.e. evaluate

$$a[n] * b[n].$$

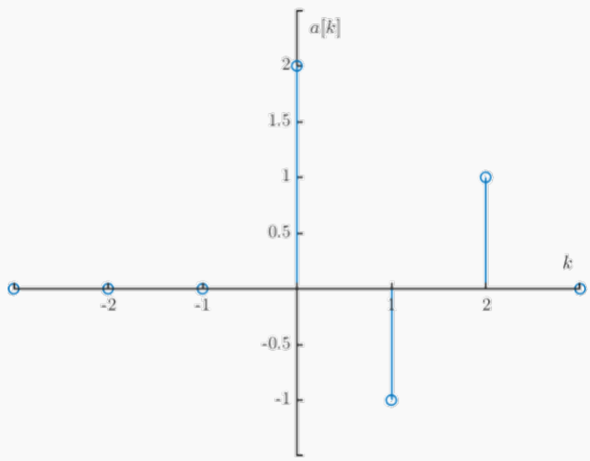
a) graphically using the flip and slide method,

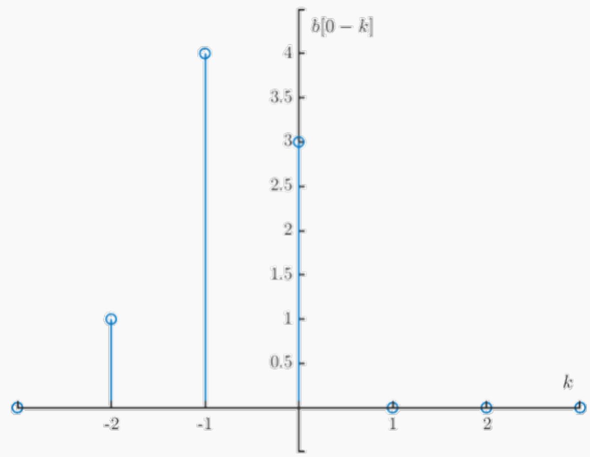
b) and using the convolution array method.

Which method do you prefer?

Exercise 5 Solution

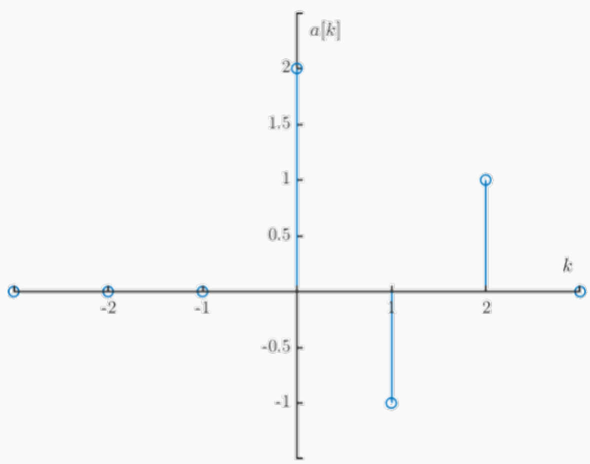
a) Graphically, we see that computing $y[0]$ looks like:

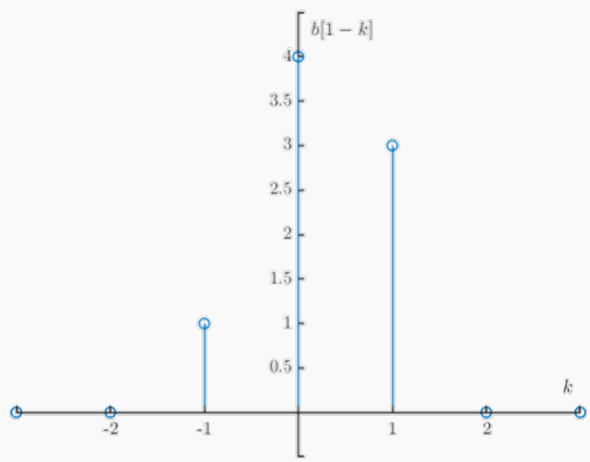




and we find that $y[0] = 2 \cdot 3 = 6$.

Similarly, we see that computing $y[1]$ looks like:





and we find that $y[1] = 2 \cdot 4 + (-1 \cdot 3) = 5$.

Continuing this process we find that

$$y[n] = \begin{bmatrix} 6 & 5 & 1 & 3 & 1 \end{bmatrix}.$$

b) Setting up the convolution array we have:

	2	-1	1
3	6	-3	3
4	8	-4	4
1	2	-1	1

and we find that

$$\begin{aligned} y[n] &= \begin{bmatrix} 6 & (8-3) & (2-4+3) & (-1+4) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 5 & 1 & 3 & 1 \end{bmatrix}. \end{aligned}$$